

On the Representations of Leavitt path algebras

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Abstract

Given a graph E we define E -algebraic branching systems, show their existence and how they induce representations of the associated Leavitt path algebra. We also give sufficient conditions to guarantee faithfulness of the representations associated to E -algebraic branching systems and to guarantee equivalence of a given representation (or a restriction of it) to a representation arising from an E -algebraic branching system.

1 Introduction

Leavitt path algebras have been introduced by G. Abrams and G. Aranda Pino (see [1]) in 2005, as algebraic analogues of graph C^* -algebras. Right after the definition of these algebras there was a spur of activity in the subject, as researches established their structure and found applications to various topics in algebra (see [1],[2], [3] or [7]). Two years after the definition of Leavitt path algebras, Mark Tomforde proved the analogue of the graph C^* -algebras uniqueness theorems to Leavitt path algebras and established the

relation between graph C^* -algebras and Leavitt path algebras (see [7]). We should note that neither the graph C^* -algebras nor the Leavitt path algebras results are obviously consequences of the others. Actually it is often the case that analogue results have completely different proofs and, moreover, neither result can be seen to imply the other.

It is in the spirit above that we write this paper. Our aim is to prove analogue versions of the representation theorems (for graph C^* -algebras) in [5] and [6], that is, to show how to obtain representations of Leavitt path algebras from E-algebraic branching systems, to study these representations and to give sufficient conditions to guarantee that a representation of $L_K(E)$ is equivalent to a representation arising from an E-algebraic branching system.

As it is often the case, we use many different techniques from the ones used in [5] and [6], and what is even more interesting, we are able to obtain deeper versions, for Leavitt path algebras, of the results in [5] and [6]. Namely, we are able to state a sufficient condition to guarantee faithfulness of a representation induced by an E-algebraic branching system (We note that our condition is still valid even in the case of a graph E that does not satisfy condition (L), in which case the Cuntz-Krieger Uniqueness theorem of [7] fail).

Among other things, we expect that the concrete faithful representations of Leavitt path algebras that we present here will deepen, and at the same time make it easier, the understanding of these algebras. Furthermore, we expect our results in the equivalence of representations to be useful for the study of irreducible representations of Leavitt path algebras.

The paper is organized as follows: Below we recall some basic terminology and definitions about Leavitt path algebras, following [7]. We devote section 2 to the introduction of E-algebraic branching systems and the representations of $L_K(E)$ induced by them. In section 3 we show that, for any graph E , we may always find representations induced by E-algebraic branching systems. We present one of the main results of the paper in section 4, where we show that for any graph with no sinks it is possible to construct faithful representations arising from E-algebraic branching systems. In order to do so, we also present a sufficient condition for a representation arising

from a E-algebraic branching system to be faithful. In section 5, we make precise what we mean by equivalence of representations, and give a sufficient condition to guarantee that a given representation (or a restriction of it) is equivalent to a representation arising from an E-algebraic branching system. Finally, in section 6, we show that for certain graphs the sufficient condition of section 5 is always satisfied, that is, any representation (or a restriction of it) of $L_K(E)$ is equivalent to a representation arising from an E-algebraic branching system.

Before we proceed, let us recall some definitions:

By a graph we always mean a directed graph $E = (E^0, E^1, r, s)$, where E^0 is a countable set of vertices, E^1 is a countable set of edges and $r, s : E^1 \rightarrow E^0$ are the range and source maps. A path is a sequence $\alpha := e_1 e_2 \dots e_n$ of edges with $r(e_i) = s(e_{i+1})$, for $1 \leq i \leq n-1$ and we say that the path α has length $|\alpha| := n$. We denote the set of paths of length n by E^n and consider the vertices in E^0 to be paths of length zero. We also let $E^* := \cup_{n=0}^{\infty} E^n$ denote the paths of finite length.

We let $(E^1)^*$ denote the set of formal symbols $\{e^* : e \in E^1\}$ and for $\alpha := e_1 e_2 \dots e_n \in E^n$ we define $\alpha^* := e_n^* e_{n-1}^* \dots e_1^*$. We also define $v^* = v$ for all $v \in E^0$.

Definition 1.1. (As in [7]). Let E be a directed graph, and K be a field. The Leavitt path algebra of E with coefficients in K , denoted $L_K(E)$, is the universal K -algebra generated by a set $\{v : v \in E^0\}$, of pairwise orthogonal idempotents, together with a set $\{e, e^* : e \in E^1\}$ of elements satisfying:

1. $s(e)e = er(e) = e$ for all $e \in E^1$
2. $r(e)e^* = e^*s(e) = e^*$ for all $e \in E^1$
3. $e^*f = \delta_{e,f}r(e)$ for all $e, f \in E^1$
4. $v = \sum_{e \in E^1: s(e)=v} ee^*$ for every vertex v with $0 < \#\{e : s(e) = v\} < \infty$

2 E-algebraic branching systems

In this section we will define E-algebraic branching systems associated to a directed graph E and we will show how these E-algebraic branching systems induce representations of the associated Leviatt path algebra, in the K algebra of the homomorphisms in a certain module.

We start with the definition of an E-algebraic branching system:

Definition 2.1. *Let X be set and let $\{R_e\}_{e \in E^1}$, $\{D_v\}_{v \in E^0}$ be families of subsets of X such that:*

1. $R_e \cap R_d = \emptyset$ for each $d, e \in E^1$ with $d \neq e$,
2. $D_u \cap D_v = \emptyset$ for each $u, v \in E^0$ with $u \neq v$,
3. $R_e \subseteq D_{s(e)}$ for each $e \in E^1$,
4. $D_v = \bigcup_{e: s(e)=v} R_e$ if $0 < \#\{e \in E^1 : s(e) = v\} < \infty$,
5. for each $e \in E^1$, there exists a bijective map $f_e : D_{r(e)} \rightarrow R_e$.

A set X , with families of subsets $\{R_e\}_{e \in E^1}$, $\{D_v\}_{v \in E^0}$, and maps f_e as above, is called an E - algebraic branching system, and we denote it by $(X, \{R_e\}_{e \in E^1}, \{D_v\}_{v \in E^0}, \{f_e\}_{e \in E^1})$, or when no confusion arises, simply by X .

Next, fix an E -algebraic branching system X . Let M be the K module of all functions from X taking values in K and let $Hom_K(M)$ denote the K algebra of all homomorphisms from M to M (with multiplication given by composition of homomorphisms and the other operations given in the usual way).

Now, for each $e \in E^1$ and for each $v \in E^0$, we will define homomorphisms S_e , S_e^* and P_v in $Hom_K(M)$.

Let S_e be defined as follows:

$$(S_e \phi)(x) = \begin{cases} \phi(f_e^{-1}(x)), & \text{if } x \in R_e \\ 0, & \text{if } x \notin R_e \end{cases},$$

where ϕ is a function in M .

In order to simplify notation, in what follows we will make a small abuse of the characteristic function symbol and denote the above homomorphism by:

$$S_e \phi = \chi_{R_e} \cdot \phi \circ f_e^{-1}.$$

In a similar fashion to what is done above, and making the same abuse of the characteristic function symbol, we define the homomorphism S_e^* by

$$S_e^* \phi = \chi_{D_{r(e)}} \cdot \phi \circ f_e,$$

where $\phi \in M$.

Finally, for each $v \in E^0$, and for $\phi \in M$, we define P_v by

$$P_v \phi = \chi_{D_v} \cdot \phi,$$

that is, P_v is the multiplication operator by χ_{D_v} , the characteristic function of D_v .

Theorem 2.2. *Let X be an E - algebraic branching system. Then there exists a representation (that is, an algebra homomorphism) $\pi : L_K(E) \rightarrow \text{Hom}_K(M)$ such that*

$$\pi(e) = S_e, \quad \pi(e^*) = S_e^* \text{ and } \pi(v) = P_v,$$

for each $e \in E^1$ and $v \in E^0$.

Proof. Since $L_K(E)$ is an universal object, all we need to do is show that the families $\{S_e, S_e^*\}_{e \in E^1}$ and $\{P_v\}_{v \in E^0}$ satisfy the relations given in definition 1.1.

It is clear that all P_v are idempotents, and orthogonality follows from item 2 in definition 2.1.

Now, let $\phi \in M$. Notice that,

$$P_{s(e)} S_e(\phi) = \chi_{D_{s(e)}} \cdot S_e(\phi) = \chi_{D_{s(e)}} \cdot \chi_{R_e} \cdot \phi \circ f_e^{-1} = S_e(\phi),$$

where the last equality follows from condition 3 in definition 2.1. In a similar way, one shows that $S_e P_{r(e)} = S_e$ and we have relation 1 in 1.1. Relation 2 of the definition of the Leviatt path algebras follows analogously.

To see that relation 3 holds notice that

$$S_e^* S_g(\phi) = \chi_{D_{r(e)}} \cdot (S_g(\phi) \circ f_e) = \chi_{D_{r(e)}} \cdot \chi_{R_g} \circ f_e \cdot \phi \circ f_g^{-1} \circ f_e = \delta_{e,g} P_{r(e)},$$

where we used that $R_e \cap R_g = \emptyset$, for $g \neq e$, to obtain the last equality.

Finally, notice that if $0 < \{e \in E^1 : s(e) = v\} < \infty$ then $D_v = \bigcup_{e:s(e)=v} R_e$, and hence

$$\begin{aligned} \sum_{\{e:s(e)=v\}} S_e S_e^*(\phi) &= \sum_{\{e:s(e)=v\}} \chi_{R_e} \cdot S_e^*(\phi) \circ f_e^{-1} = \sum_{\{e:s(e)=v\}} \chi_{R_e} \cdot \chi_{D_{r(e)}} \circ f_e^{-1} \cdot \phi = \\ &= \sum_{\{e:s(e)=v\}} \chi_{R_e} \cdot \phi = \chi_{D_v} \cdot \phi = P_v(\phi). \end{aligned}$$

□

Remark 2.3. Notice that theorem 2.2 still holds if we change the module M of all functions from X to K for the module of all functions from X to K that vanish in all, but a finite number of points, of X .

In the next section we consider the question of existence of E-algebraic branching systems (and their induced representations) for any given graph E .

3 Existence of E-algebraic branching systems

Let E be a graph, with E^0 and E^1 countable. Next we show that there exists an E-algebraic branching system in \mathbb{R} associated. Our proof is constructive and one can actually obtain a great number of E-algebraic branching systems following the ideas below.

Theorem 3.1. Let $E = (E^0, E^1, r, s)$ be a graph, with E^0, E^1 both countable. Then there exists an E -branching system X , where X is an (possible unlimited) interval of \mathbb{R} .

Proof. Let $E^1 = \{e_i\}_{i=1}^\infty$ (or, if E^1 is finite, let $E^1 = \{e_i\}_{i=1}^N$). For each $i \geq 1$ define $R_{e_i} = [i-1, i)$. Let $W = \{v \in E^0 : v \text{ is a sink}\}$ (a vertex $v \in E^0$ is a sink if $v \notin s(E^1)$). Note that W is finite or infinite countable. Write $W = \{v_i : i = 1, 2, 3, \dots\}$. For each $v_i \in W$, define $D_{v_i} = [-i, -i+1)$. For the vertices $u \in E^0$ which are not sinks, define $D_u = \bigcup_{e_i: s(e_i)=u} R_{e_i}$. Note that items 1-4 from definition 2.1 are satisfied. It remains to define functions which satisfy item 5.

Let $\bar{e} \in E^1$.

If $r(\bar{e})$ is a sink then $r(\bar{e}) = v_i \in W$, and so $D_{r(\bar{e})} = [-i, -i+1)$. Then we define $f_{\bar{e}} : D_{r(\bar{e})} \rightarrow R_{\bar{e}}$ as any bijection between these sets (for example, the linear bijection).

If $r(\bar{e}) = \bar{v}$ is not a sink, then

$$D_{r(\bar{e})} = D_{\bar{v}} = \bigcup_{e: s(e)=\bar{v}} R_e.$$

To define the function $f_{\bar{e}} : D_{r(\bar{e})} \rightarrow R_{\bar{e}}$ in this case we proceed as follows.

First we divide the interval $R_{\bar{e}}$ in $\#\{e : s(e) = \bar{v}\}$ pairwise disjoint (open on the right and closed on the left hand side) intervals I_e (notice that we might have to divide $R_{\bar{e}}$ in a countable infinite number of intervals). Then, we define $\tilde{f}_{\bar{e}} : \bigcup_{e: s(e)=\bar{v}} R_e \rightarrow \bigcup_{e: s(e)=\bar{v}} I_e$ so that $\tilde{f}_{\bar{e}}|_{R_e}$ is a bijection between R_e and I_e (for example, the linear bijection).

Now, defining

$$X = \left(\bigcup_{e_i \in E^1} R_{e_i} \right) \cup \left(\bigcup_{v_i \in W} D_{v_i} \right)$$

we obtain the desired E-algebraic branching system. □

Theorem 3.1 together with theorem 2.2 guarantees that every Leavitt path algebra $L_K(E)$ of a countable graph E may be represented in $\text{Hom}_K(M)$. Let us summarize this result in the following corollary:

Corollary 3.2. *Given a countable graph E , there exists a homomorphism*

$\pi : L_K(E) \rightarrow \text{Hom}_K(M)$ such that

$$\pi(v)(\phi) = \chi_{D_v} \cdot \phi, \quad \pi(e)(\phi) = \chi_{R_e} \cdot \phi \circ f_e^{-1} \quad \text{and} \quad \pi(e^*)(\phi) = \chi_{D_{r(e)}} \cdot \phi \circ f_e$$

for each $\phi \in M$, where M is the K module of all functions from X taking values in K , X is an (possible unlimited) interval of \mathbb{R} , and R_e and D_v are as in theorem 3.1

We now seek conditions that guarantee the faithfulness of the representations we have constructed above (of course when the Leavitt path algebra is simple any non-zero representation is faithful).

4 Faithful representations of Leavitt path algebras of row-finite graphs without sinks

Important results regarding faithfulness of a representation in the literature include the Graded Uniqueness theorem and the Cuntz-Krieger Uniqueness theorem (see [7]). In fact, we may use the Cuntz-Krieger Uniqueness theorem for the representations of corollary 3.2. This theorem guarantees that for any graph E that satisfies condition (L) (each closed path in E has an exit, that is, if $\alpha = \alpha_1 \dots \alpha_n \in E^n$ with $s(\alpha) = r(\alpha)$, then there exists $e \in E$ such that $s(e) = s(\alpha_i)$ and $e \neq e_i$ for some i) faithfulness of a representation follows simply by checking that the representation does not vanish at the vertices of the graph. This follows promptly for the representations of corollary 3.2 and hence, for graphs that satisfy condition (L), they are faithful.

As we could see above, the Cuntz-Krieger Uniqueness theorem is a very powerful tool, but it excludes some very simple examples, as for the graph E defined by $E^0 = \{*\}$ and $E^1 = \{x\}$ (E consists of one vertex and one "loop" edge). The Leavitt path algebra associated to this graph is $K[x, x^1]$, the Laurent polynomials algebra, see [1].

In order to overcome problems as the one mentioned above, in this section we introduce a sufficient condition (valid for row finite graphs without

sinks) to guarantee that a representation of $L_K(E)$ induced by an E-algebraic branching system is faithful.

Recall that a graph is row-finite if $s^{-1}(v)$ is finite, for each $v \in E^0$, and a sink is a vertex which emits no edges.

Let $(X, \{R_e\}_{e \in E^1}, \{D_v\}_{v \in E^0}, \{f_e\}_{e \in E^1})$ be an E -algebraic branching system. A closed path $\alpha = e_1 \dots e_n$ in the graph E is a path such that $r(e_i) = s(e_{i+1})$ and $r(\alpha) := r(e_n) = v = s(e_1) =: s(\alpha)$. For a closed path α , let

$$f_\alpha : D_v \rightarrow R_{e_1} \subseteq D_v$$

denote the composition

$$f_\alpha := f_{e_1} \circ \dots \circ f_{e_n}.$$

Remark 4.1. Notice that since α is a path f_α is well defined.

Theorem 4.2. Let $(X, \{R_e\}_{e \in E^1}, \{D_v\}_{v \in E^0}, \{f_e\}_{e \in E^1})$ be an E -algebraic branching system for a row-finite graph without sinks E . Suppose that for each finite set of closed paths $\{\alpha^1, \dots, \alpha^n\}$ in E , beginning on the same vertex v , there is an element $z_0 \in D_v$ such that $f_{\alpha^i}(z_0) \neq z_0$ for all $i \in \{1, \dots, n\}$. Then, the representation of $L_K(E)$ induced by this E -algebraic branching system is faithful.

Proof. Let $\pi : L_K(E) \rightarrow \text{Hom}_K(M)$ be the representation induced by the E-algebraic branching system, as in theorem 2.2. (Recall that M is the K -module of functions from X to K).

Let $x \in L_K(E)$, $x \neq 0$. Our aim is to show that $\pi(x) \neq 0$. We will separate the proof in a few steps. We start with:

Step 1: For each $n \in \mathbb{N}$, there exists a path $e_1 \dots e_n$ of length n such that $xe_1 \dots e_n \neq 0$.

First, write x as a finite sum,

$$x = \sum_{i \in I} \lambda_i \alpha_i \beta_i^* + \sum_{j \in J} \lambda_j \alpha_j + \sum_{k \in L} \lambda_k v_k,$$

where $\alpha_i, \beta_i, \alpha_j$ are paths, $v_k \in E^0$, $\lambda_i, \lambda_j, \lambda_k \in K$, $|\beta_i| \geq 1$ for each $i \in I$,

and $|\alpha_j| \geq 1$ for all $j \in J$ (observe that this can be done by Corollary 3.2 in [7]).

Consider the set of vertices $\{s(\beta_i) : i \in I\} \cup \{r(\alpha_j) : j \in J\} \cup \{v_k : k \in K\}$, and write this set as $\{u_1, \dots, u_N\}$ where $u_i \neq u_j$ for $i \neq j$. Also, notice that $\sum_{i=1}^N u_i$ is a right identity for x , that is $x \sum_{i=1}^N u_i = x$.

Now, suppose that $xe = 0$ for all $e \in E^1$. Since E is row-finite and has no sinks, then $u_i = \sum_{e \in s^{-1}(u_i)} ee^*$ for all i , and hence

$$x = x \sum_{i=1}^N u_i = x \sum_{i=1}^N \sum_{e \in s^{-1}(u_i)} ee^* = 0,$$

which contradicts the fact that $x \neq 0$.

So, it follows that there exists an edge e_1 such that $xe_1 \neq 0$. Now, given $n \in \mathbb{N}$, we may proceed recursively and obtain a path $e_1 e_2 \dots e_n$ such that $xe_1 \dots e_{n-1} e_n \neq 0$ as desired.

In order to state our next step we need to make a few observations.

First, notice that, from step 1, we may find an n sufficiently large such that the product $xe_1 \dots e_n$ may be written as a finite sum:

$$xe_1 \dots e_n = \sum_{i=1}^p \gamma_i c^i \neq 0,$$

where c^i are paths in E with $|c^i| \geq 1$ for all i , $\gamma_i \neq 0$ for all i , and $c^i \neq c^j$ for $i \neq j$.

Also, for each $z \in \bigcup_{u \in E^0} D_u$, denote by δ_z the function defined by $\delta_z(y) = [y = z]$, where $[y = z] = 0$ if $y \neq z$ and $[y = z] = 1$ if $y = z$.

In the next step we characterize how $\pi(e_1 \dots e_n)$ acts on δ_z .

Step 2: For each path $d_1 \dots d_n$ in E and each δ_z we have that

$$\pi(d_1 \dots d_n)(\delta_z) = [z \in D_{r(d_n)}] \delta_{f_{d_1 \dots d_n}(z)}.$$

The proof of this step follows from the fact that $\pi(e)(\phi) = \chi_{R_e} \cdot \phi \circ f_e^{-1}$

for each edge, and is left to the reader.

Next, fix $c \in \{c^1, \dots, c^p\}$ such that $|c| \leq |c^i|$ for all $i \in \{1, \dots, p\}$. Notice that, from Step 2, $\pi(c)(\delta_z) = \delta_{f_c(z)}$ for all $z \in D_{r(c)}$, and we have:

Step 3: Let α be a path in E . If $r(\alpha) \neq r(c)$ then $\pi(\alpha)(\delta_z) = 0$ for all $z \in D_{r(c)}$.

To see this, just notice that if $r(c) \neq r(\alpha)$ then $D_{r(\alpha)} \cap D_{r(c)} = \emptyset$. So, for $z \in D_{r(c)}$, by Step 2 it follows that $\pi(\alpha)\delta_z = 0$.

Step 4: For all $c^i \in \{c^1, \dots, c^p\} \setminus \{c\}$ with $|c^i| = |c|$ we have that

$$\pi(c^i)(\delta_z)(f_c(z)) = 0$$

for all $z \in D_{r(c)}$.

The proof of this step goes as follows:

Write $c^i = d_1 \dots d_n$ and $c = c_1 \dots c_n$. Let j_0 be the smallest of the indexes j such that $d_j \neq c_j$, and let $z \in D_{r(c)}$.

We claim that $f_c(z) \neq f_{c^i}(z)$. Suppose not. Then

$$f_{c_1} \circ \dots \circ f_{c_{j_0-1}} \circ f_{c_{j_0}} \circ \dots \circ f_{c_n}(z) = f_{c_1} \circ \dots \circ f_{c_{j_0-1}} \circ f_{d_{j_0}} \circ \dots \circ f_{d_n}(z)$$

and since the f_e 's are bijections for all $e \in E^1$, the above equality implies that

$$f_{c_{j_0}} \circ \dots \circ f_{c_n}(z) = f_{d_{j_0}} \circ \dots \circ f_{d_n}(z),$$

which is a contradiction, since $c_{j_0} \neq d_{j_0}$ implies that the image of $f_{c_{j_0}}$ is disjoint from the image of $f_{d_{j_0}}$.

So, $f_c(z) \neq f_{c^i}(z)$, and hence

$$\pi(c^i)(\delta_z)(f_c(z)) = \delta_{f_{c^i}(z)}(f_c(z)) = 0.$$

Step 5: There exists $z_0 \in D_{r(c)}$ such that $\pi(c^i)(\delta_{z_0})(f_c(z_0)) = 0$ for all $c^i \in$

$\{c^1, \dots, c^p\} \setminus \{c\}$.

First notice that, by steps 3 and 4, if $r(c^i) \neq r(c)$, or if $|c^i| = |c|$ and $c^i \neq c$, then $\phi(c^i)(\delta_z) = 0$ for all $z \in D_{r(c)}$.

So, let c^i be such that $|c^i| > |c|$ and $r(c^i) = r(c)$. Write $c = c_1 \dots c_n$ and $c^i = d_1 \dots d_n \dots d_m$. If $c_j \neq d_j$, for some $j \in \{1, \dots, n\}$, then proceeding analogously as in the proof of step 4, we have that $f_c(z) \neq f_{c^i}(z)$ for all $z \in D_{r(c)}$, and hence $\pi(c^i)(\delta_z)(f_c(z)) = \delta_{f_{c^i}(z)}(f_c(z)) = 0$ in this case.

We are left with the case when c^i is an element of W , where $W \subseteq \{c^1, \dots, c^p\} \setminus \{c\}$ is defined by $W = \{c^i : c^i = c\alpha^i \text{ and } r(c^i) = r(c)\}$. Notice that each α^i is a closed path from $r(c)$ to $r(c)$. By the hypothesis of the theorem, there exists $z_0 \in D(r(c))$ such that $f_{\alpha^i}(z_0) \neq z_0$ for all α_i . Then,

$$f_{c^i}(z_0) = f_c \circ f_{\alpha^i}(z_0) \neq f_c(z_0)$$

for all $c^i \in W$ and step 5 is proved.

Let us now conclude the proof of this theorem. Recall that we started with a $x \in L_K(E)$, $x \neq 0$, and have considered the element

$$xe_1 \dots e_n = \sum_{i=1}^p \gamma_i c^i,$$

where $\gamma_i \neq 0$ and $c^i \neq c^j$ for $i \neq j$.

By the previous steps, there exists a $c \in \{c^1, \dots, c^p\}$ and a $z_0 \in D_{r(c)}$ such that $\pi(c^i)(\delta_{z_0})(f_c(z_0)) = 0$ for all $c^i \in \{c^1, \dots, c^p\} \setminus \{c\}$. But then,

$$\begin{aligned} (\pi(x)\pi(e_1 \dots e_n))(\delta_{z_0})(f_c(z_0)) &= \pi(xe_1 \dots e_n)(\delta_{z_0})(f_c(z_0)) = \\ &= \sum_{i=1}^p \gamma_i \pi(c^i)(\delta_{z_0})(f_c(z_0)) = \gamma_{i_0} \delta_{f_c(z_0)}(f_c(z_0)) = \gamma_{i_0}, \end{aligned}$$

where i_0 is such that $c = c^{i_0}$.

So, it follows that $\pi(x)\pi(e_1 \dots e_n) \neq 0$, and hence $\pi(x) \neq 0$. \square

Theorem 4.2 is an important result. It allow us to construct faithful representations of $L_K(E)$, when E is a row finite graph without sinks. For these

graphs we describe E-algebraic branching systems that satisfy the conditions of theorem 4.2 (and hence induce faithful representations) below.

So, let E be a countable row-finite graph without sinks.

Since E^1 is finite or infinite countable we have that $E^1 = \{e_1, e_2, e_3, \dots, e_N\}$ or $E^1 = \{e_1, e_2, e_3, \dots\}$. For each $e_i \in E^1$ define $R_{e_i} := [i-1, i)$. For a vertex $v \in E^0$ define $D_v := \bigcup_{e \in s^{-1}(v)} R_e$. We also need to define bijective maps $f_e : D_{r(e)} \rightarrow R_e$, for each $e \in E^1$. To do this, first fix an irrational number $\theta \in [0, 1)$, and let $h_\theta : [0, 1) \rightarrow [0, 1)$ be defined by $h_\theta(x) = (x + \theta) \bmod(1)$, which is a bijective map. For $a, b \in \mathbb{R}$ with $a < b$, define $g_a^b : [0, 1) \rightarrow [a, b)$ by $g_a^b(x) = bx + (1-x)a$, which is also a bijective map, with inverse $(g_a^b)^{-1}$.

Let $e \in E^1$.

Since $r(e)$ is not a sink and E is row-finite then $0 < \#\{s^{-1}(r(e))\} < \infty$. So, $s^{-1}(r(e)) = \{e_{i_1}, \dots, e_{i_P}\}$ and hence

$$D_{r(e)} = \bigcup_{k=1}^P R_{e_{i_k}}.$$

Since $e = e_j$, for some $j \in \mathbb{N}$, we have that $R_e = [j-1, j)$. Write R_e as the following disjoint union:

$$R_e = [j-1, j) = \bigcup_{k=1}^P \left[j-1 + \frac{k-1}{P}, j-1 + \frac{k}{P} \right).$$

Now, given $x \in D_{r(e)}$, we have that $x \in R_{e_{i_k}} = [i_k-1, i_k)$ for some $k \in \{1, \dots, P\}$, and we define

$$f_e(x) := \left(g_{(j-1+\frac{k-1}{P})}^{(j-1+\frac{k}{P})} \circ h_\theta \circ \left(g_{(i_k-1)}^{i_k} \right)^{-1} \right) (x),$$

that is, f_e restricted to $R_{e_{i_k}}$ is the composition

$$R_{e_{i_k}} \xrightarrow{\left(g_{(i_k-1)}^{i_k} \right)^{-1}} [0, 1) \xrightarrow{h_\theta} [0, 1) \xrightarrow{g_{(j-1+\frac{k-1}{P})}^{(j-1+\frac{k}{P})}} \left[j-1 + \frac{k-1}{P}, j-1 + \frac{k}{P} \right).$$

This defines $f_e : D_{r(e)} \rightarrow R_e$ as a bijective map, and it is not hard to see

that $f_e(x) = \frac{x+\theta+r_e(x)}{P}$, where $r_e(x)$ is a rational number, for each $x \in D_r(e)$.

So, for each $e \in E^1$, we have defined a bijective map $f_e : D_{r(e)} \rightarrow R_e$, such that

$$f_e(x) = \frac{x + \theta + r_e(x)}{P_e},$$

where $r_e(x)$ is a rational number, for each $x \in D_{r(e)}$ and P_e is a natural number, namely $P_e = \#\{s^{-1}(r(e))\}$.

Defining $X = \bigcup_{e \in E^1} R_e$ we obtain an E -algebraic branching system

$$(X, \{D_u\}_{u \in E^0}, \{R_e\}_{e \in E^1}, \{f_e\}_{e \in E^1}),$$

and hence we obtain a representation $\pi : L_K(E) \rightarrow \text{Hom}_K(M)$ (as in theorem 2.2).

Corollary 4.3. *Let E be a row finite graph with no sinks. Then the representation $\pi : L_K(E) \rightarrow \text{Hom}_K(M)$ induced by the E -algebraic branching system constructed above is faithful.*

Proof. All we need to do is verify the hypothesis of theorem 4.2, that is, we need to check that for each finite set $\{\alpha^1, \dots, \alpha^N\}$ of closed paths beginning on the same vertex v , there exists an element $z_0 \in D_v$ such that $f_{\alpha^i}(z_0) \neq z_0$ for all $i \in \{1, \dots, N\}$.

So, let $\alpha = c_1 \dots c_n$ be a closed path beginning on v . Notice that, for each $x \in D_v$

$$f_{c_1} \circ \dots \circ f_{c_n}(x) = \frac{x}{P_{c_1} \dots P_{c_n}} + \theta \left(\frac{1}{P_{c_1} \dots P_{c_n}} + \frac{1}{P_{c_1} \dots P_{c_{n-1}}} + \dots + \frac{1}{P_{c_1}} \right) + \bar{r}(x),$$

where $\bar{r}(x)$ is a rational number and P_{c_1}, \dots, P_{c_n} are natural numbers. It follows that, if $x \in D_v$ is a rational number, then $f_{c_1} \circ \dots \circ f_{c_n}(x)$ is an irrational number and hence no rational number is a fixed point for f_α . Then, for any finite set $\{\alpha^1, \dots, \alpha^N\}$ of closed paths in E beginning on v , we may choose $z_0 \in D_v$ to be a rational number, and so $f_{\alpha^i}(z_0) \neq z_0$ for all $i \in \{1, \dots, N\}$ as desired. \square

Example 4.4. *Let $E^0 = \{*\}$, $E^1 = \{x\}$ as in the figure below.*



Notice that $L_K(E) = K[x, x^{-1}]$, the Laurent polynomials in x and x^{-1} . By corollary 4.3 above, the representation induced by the E-algebraic branching system X , where $R_e = [0, 1]$, $D_{r(e)} = [0, 1]$ and $f_e : D_{r(e)} \rightarrow R_e$ is defined by $f_e(x) = x + \theta \bmod 1$ (that is, f_e is rotation by an irrational number θ) is faithful. It follows that the K algebra of the Laurent polynomials in x and x^{-1} is isomorphic to the sub algebra of $\text{Hom}_K(M)$ generated by $\{S_e, S_e^*\}$, where, for $f \in M$, $S_e \phi = \phi \circ f_e^{-1}$ and $S_e^* \phi = \phi \circ f_e$.

5 Equivalence of representations of $L_K(E)$

In the previous sections, we have introduced a class of representations of the Leavitt path algebras induced by E-algebraic branching systems. One question which remains is if any representation may be obtained in such a manner.

In this section, we show that under a certain condition over a graph E , each K -algebra homomorphism $\tilde{\pi} : L_K(E) \rightarrow A$ has a sub-representation associated to it which is equivalent to a representation induced by an E -algebraic branching system.

Before we proceed, notice that given a K -algebra A , there exist a K -module V and an injective K -algebra homomorphism $\varphi : A \rightarrow \text{Hom}_K(V)$. To see this, note that $A \times K$ is a unital K -algebra, with the operations defined by $(a, k) + (b, l) := (a + b, k + l)$, $k(a, l) := (ka, kl)$ and $(a, k)(b, l) := (ab + la + kb, kl)$ for each $a, b \in A$ and $k, l \in K$. In particular, $V := A \times K$ is a K -module. Defining, for each $a \in A$, $\varphi(a) : V \rightarrow V$ by $\varphi(a)(b, k) = (a, 0)(b, k)$, we obtain an injective homomorphism $\varphi : A \rightarrow \text{Hom}_K(V)$.

Given a K -algebra homomorphism $\tilde{\pi} : L_K(E) \rightarrow A$, using the previous injective K -algebra homomorphism φ , we may consider the composition $\varphi \circ \tilde{\pi} : L_K(E) \rightarrow \text{Hom}_K(V)$. With this in mind, from now on, we only con-

sider representations (K -algebra homomorphism) from $L_K(E)$ to $\text{Hom}_K(V)$, where V is a K -module.

Next we will prepare the ground for the results in this section. We start with a representation $\Phi : L_K(E) \rightarrow \text{Hom}_K(V)$ and define K -submodules

$$V_u = \Phi(u)(V)$$

and

$$V_e = \Phi(e)\Phi(e^*)(V),$$

for all $u \in E^0$ and all $e \in E^1$. Since Φ is a representation of $L_K(E)$, it satisfies the relations of Definition 1.1, and it follows that:

1. $V_e \subseteq V_{s(e)}$ for each $e \in E^1$,
2. $V_e \cap V_f = 0$ for each $e, f \in E^1$, $e \neq f$,
3. $V_u \cap V_w = 0$ for each $u, w \in E^0$, $u \neq w$,
4. $\Phi(e) : V_{r(e)} \rightarrow V_e$ is a K -module isomorphism, with inverse $\Phi(e^*)$,
5. $V_u = \bigoplus_{e:s(e)=u} V_e$ if $0 < \#\{e : s(e) = u\} < \infty$
6. $V_u = \left(\bigoplus_{e:s(e)=u} V_e \right) \oplus \overline{V}_u$ if $\#\{e : s(e) = u\} = \infty$, where \overline{V}_u is some K -submodule of V_u ,
7. $V = \left(\bigoplus_{u \in E^0} V_u \right) \oplus \overline{V}$, where \overline{V} is a K -submodule of V .

To obtain the equality of item 6 above, notice that V_u is a K -vector space, and hence we may complete the (Hammel) basis of $\bigoplus_{e:s(e)=u} V_e$ to obtain a basis of V_u . The same holds for the last equality.

We now intend to pick a particular basis for the K -vector space V . By equality 7 above, we need to choose a basis for V_u , $u \in E^0$, and \overline{V} .

Before picking the basis for V_u , notice that, since V_e and \overline{V}_u are K -vector spaces, there exists Hammel basis $\{m_x : x \in R_e\}$ for each V_e and $\{m_x : x \in \overline{I}_u\}$ for each \overline{V}_u . Choose the index sets R_e and \overline{I}_u as being pairwise disjoint,

that is, $R_e \cap R_f = \emptyset$, $R_e \cap \overline{I}_u = \emptyset$ and $\overline{I}_u \cap \overline{I}_w = \emptyset$. Now, we define the basis of V_u in the following way:

- if $u \notin s^{-1}(E^1)$, choose some basis $\{m_x : x \in D_u\}$ of V_u , where D_u is an index set of the basis.
- if $0 < \#\{e \in E^1 : s(e) = u\} < \infty$ let $D_u := \bigcup_{R_e : s(e)=u} R_e$ and so $\{m_x : x \in D_u\}$ is a basis of V_u .
- if $\#\{e \in E^1 : s(e) = u\} = \infty$ let $D_u := \left(\bigcup_{R_e : s(e)=u} R_e \right) \cup \overline{I}_u$ and so $\{m_x : x \in D_u\}$ is a basis of V_u .

The index sets D_u obtained in the second and last items above are pairwise disjoint. In the first item, choose the index sets D_u such that the sets $\{D_u\}_{u \in E^0}$ are pairwise disjoint.

Finally, choose a basis $\{m_x : x \in \overline{I}\}$ of \overline{V} and an index set \overline{I} such that $\overline{I} \cap D_u = \emptyset$ for all $u \in E^0$. We have now chosen a basis for V .

Let

$$W = \bigoplus_{u \in E^0} V_u.$$

Recall that $V = W \oplus \overline{V}$. Let $P_1 : V \rightarrow W$ and $P_2 : V \rightarrow \overline{V}$ be the two canonical projections and $i_1 : W \rightarrow V$ and $i_2 : \overline{V} \rightarrow V$ be the two canonical inclusions. Notice that, for each $a \in L_K(E)$, it holds that

$$\Phi(a) = (P_1 + P_2)\Phi(a)(i_1 \oplus i_2) = P_1\Phi(a)i_1 + P_2\Phi(a)i_2 = P_1\Phi(a),$$

since $P_2\Phi(a) = 0$. So, we will consider the "restriction of Φ to W ", that is, the map

$$\begin{aligned} \Phi_1 : L_K(E) &\rightarrow \text{Hom}_K(W) \\ a &\mapsto P_1\Phi(a)i_1 = \Phi(a)i_1 \end{aligned},$$

which is a representation.

Our aim is to show (under some additional hypothesis) that the representation Φ_1 is equivalent, in some sense, to a representation induced by an E -algebraic branching system. So, we need to define the desired branching system.

Let $X = \bigcup_{u \in E^0} D_u$. By the definition of D_v and R_e , it is clear that conditions 1-4 of definition 1.1 are satisfied. To obtain an E -algebraic branching system, we need to define bijective maps $f_e : D_{r(e)} \rightarrow R_e$. Recall that the restriction $\Phi(e) : V_{r(e)} \rightarrow V_e$ is a K -module isomorphism, with inverse $\Phi(e^*)$, and the sets $D_{r(e)}$ and R_e are the index sets of the basis of $V_{r(e)}$ and V_e , respectively. So, if the basis of $D_{r(e)}$ is taken to the basis of R_e , that is, if for each $x \in D_{r(e)}$ we have that $\Phi(e)(m_x) = m_y$ for some $y \in R_e$, then the map $D_{r(e)} \ni x \mapsto y \in R_e$ defines a bijective map f_e .

So, from now on we assume this additional hypothesis, that is, we assume that:

$$\Phi(e)(\{m_x : x \in D_{r(e)}\}) = \{m_y : y \in R_e\}, \text{ for each } e \in E^1, \quad (B2B)$$

which we call condition (B2B). Notice that condition (B2B) is equivalent to say that $\Phi(e^*)(\{m_y : y \in R_e\}) = \{m_x : x \in D_{r(e)}\}$ for each $e \in E^1$.

We may now define $f_e : D_{r(e)} \rightarrow R_e$ by $f_e(x) = y$, where y is such that $\Phi(m_x) = m_y$.

Notice that the map f_e is bijective, for each $e \in E^1$, and hence the set X with the families $\{R_e\}_{e \in E^1}$, $\{D_u\}_{u \in E^0}$ and $\{f_e\}_{e \in E^1}$ is an E -algebraic branching system.

Before we state our next theorem, we need the following definition:

Definition 5.1. Let $\pi : L_K(E) \rightarrow \text{Hom}_K(M)$ and $\Phi : L_K(E) \rightarrow \text{Hom}_K(W)$ be representations of $L_K(E)$, where M and W are K -modules. We say that π is equivalent to ϕ if there exists a K -module isomorphism $U : W \rightarrow M$ such that the diagram

$$\begin{array}{ccc} W & \xrightarrow{\Phi_1(a)} & W \\ U \downarrow & & \downarrow U \\ M & \xrightarrow{\pi(a)} & M \end{array}$$

commutes, for each $a \in L_K(E)$, and there is a K -algebra isomorphism $T : \text{Hom}_K(W) \rightarrow \text{Hom}_K(M)$ such that the diagram

$$\begin{array}{ccc} L_K(E) & \xrightarrow{\Phi_1} & \text{Hom}_K(W) \\ \text{id} \downarrow & & \downarrow T \\ L_K(E) & \xrightarrow{\pi} & \text{Hom}_K(M) \end{array}$$

commutes.

For what follows, let $M := \{g : X \rightarrow K : g(x) \neq 0 \text{ only for finitely many } x \in X\}$, $Y = X \cup \bar{I}$ (recall that \bar{I} is the index set of \bar{V}) and $N = \{g : Y \rightarrow K : g(x) \neq 0 \text{ only for finitely many } x \in Y\}$. Recall that $W = \bigoplus_{u \in E^0} V_u$. We are now ready to prove the next theorem.

Theorem 5.2. *Let $\Phi : L_K(E) \rightarrow \text{Hom}_K(V)$ be a representation. Choose a basis of (the K -vector space) V as constructed above. Suppose that this basis satisfies condition (B2B). Suppose also that $\Phi(e^*)(\overline{V_{s(e)}}) = 0$, for all $e \in E^1$, where $\overline{V_{s(e)}}$ was defined in item 6 above. Then:*

1. *There exists a representation $\pi : L_K(E) \rightarrow \text{Hom}_K(M)$, induced by an E -algebraic branching system, which is equivalent to Φ_1 (the restriction of Φ to W).*
2. *if \bar{V} (as in item 7 above) may be chosen such that $\Phi(u)(\bar{V}) = 0$, for each $u \in E^0$, then there exists an E -algebraic branching system which induces a representation $\pi : L_K(E) \rightarrow \text{Hom}_K(N)$ which is equivalent to Φ .*

Proof. We begin by proving the first part. Let $(X, \{R_e\}_{e \in E^1}, \{D_u\}_{u \in E^0}, \{f_e\}_{e \in E^1})$ and M be as defined in the paragraphs preceding this theorem. By theorem 2.2, there exists a representation $\pi : L_K(E) \rightarrow \text{Hom}_K(M)$ such that $\pi(e)(g) = \chi_{R_e} \cdot g \circ f_e^{-1}$, $\pi(e^*)(g) = \chi_{D_{r(e)}} \cdot g \circ f_e$ and $\pi(u)(g) = \chi_{D_u} \cdot g$.

Notice that M is a K -module with basis $\{\delta_x\}_{x \in X}$, where $\delta_x : X \rightarrow K$ is defined by $\delta_x(y) = 0$, if $y \neq x$ and $\delta_x(y) = 1$, if $y = x$.

Recall that $\{m_x : x \in X\}$ is a basis of V . So, the map $\{m_x : x \in X\} \ni m_x \mapsto \delta_x \in M$ induces a K -module isomorphism $U : W \rightarrow M$.

Next we show that $\Phi_1(a) = U^{-1} \circ \pi(a) \circ U$ for each $a \in L_K(E)$. Notice that it is enough to show that $\Phi_1(e) = U^{-1} \circ \pi(e) \circ U$ and $\Phi_1(e^*) = U^{-1} \circ \pi(e^*) \circ U$, for each $e \in E^1$ and $\Phi_1(v) = U^{-1} \circ \pi(v) \circ U$, for all $v \in E^0$. We will verify the second equality and the other two are left to the reader.

Notice that, for $x \in X$, $\pi(e^*)(\delta_x) = [x \in R_e] \delta_{f_e^{-1}(x)}$ (where $[x \in R_e] = 1$, if $x \in R_e$ and $[x \in R_e] = 0$, if $x \notin R_e$). So, it follows that $U^{-1} \circ \pi(e^*) \circ U(m_x) = [x \in R_e] \delta_x$.

We now evaluate $\Phi_1(e^*)(m_x)$. If $m_x \in V_u$, with $u \neq s(e)$, then $\Phi_1(e^*)(m_x) = \Phi(e^*)\Phi(s(e))\Phi(u)(m_x) = 0$, since $\Phi(s(e))\phi(u) = 0$. Let $m_x \in V_{s(e)}$. Recall that

$$V_{s(e)} = \left(\bigoplus_{d \in E^1: s(d)=s(e)} V_d \right) \oplus \overline{V_{s(e)}}.$$

If $m_x \in V_d$, for some $d \neq e$, then $\Phi_1(e^*)(m_x) = \Phi(e^*)\Phi(d)\Phi(d^*)(m_x) = 0$ (since $\Phi(e^*)\Phi(d) = 0$). If $m_x \in \overline{V_{s(e)}}$, then $\Phi(e^*)(m_x) = 0$ by hypothesis. It remains to evaluate $\Phi_1(e^*)(m_x)$ for $m_x \in V_e$. In this case, $\Phi_1(e^*)(m_x) = \Phi(e^*)(m_x) = m_{f_e^{-1}(x)}$, by the definition of the map f_e^{-1} .

So, it follows that $U^{-1} \circ \pi(e^*) \circ U = \Phi_1(e^*)$ as desired and we have that

$$\Phi_1(a) = U^{-1} \circ \pi(a) \circ U,$$

for all $a \in L_K(E)$.

Defining $T : \text{Hom}_K(W) \rightarrow \text{Hom}_K(M)$ by $T(A) = U \circ A \circ U^{-1}$, which is a K -algebra isomorphism, we obtain that Φ_1 is equivalent to π .

To prove the second part of the theorem consider the E -algebraic branching system $Y = X \cup \bar{I}$ (recall that \bar{I} is the index set of \bar{V}). Consider $\{R_e\}_{e \in E^1}$, $\{D_u\}_{u \in E^0}$ and $\{f_e\}_{e \in E^1}$ as in the first part. This E -algebraic branching system induces a representation $\pi : L_K(E) \rightarrow \text{Hom}_K(N)$, where

$$N = \{g : Y \rightarrow K : g(x) \neq 0 \text{ only for finitely many } x \in Y\}.$$

The map $V \ni m_x \rightarrow \delta_x \in N$ induces a K -module isomorphism $Q : V \rightarrow N$, and the map $L : \text{Hom}_K(V) \rightarrow \text{Hom}_K(N)$ defined by $L(A) = Q \circ A \circ Q^{-1}$, for each $A \in \text{Hom}_K(V)$ is also an isomorphism. The rest of the proof follows

analogously to what was done above for the first part of the theorem. \square

Remark 5.3. a) If the graph E is row-finite, then $V_{s(e)} = \bigoplus_{d \in E^1: s(d)=s(e)} V_d$, and the condition $\Phi(e^*)(\overline{V_{s(e)}}) = 0$ (which appears in the hypothesis of the previous theorem) is vacuously satisfied. So, the first part of the previous theorem applies to any representation of row-finite graphs, as long as (B2B) is satisfied.

b) If E^0 is finite then \overline{V} may be chosen so that $\Phi(u)(\overline{V}) = 0$, for each $u \in E^0$. In fact, if E^0 is finite, define

$$\overline{V} = \{m \in V : \Phi(u)(m) = 0 \ \forall u \in E^0\}.$$

Then it is clear that $\left(\bigoplus_{u \in E^0} V_u\right) \oplus \overline{V} \subseteq V$. For a given $m \in V$, write m as the sum

$$m = \left(\sum_{u \in E^0} \Phi(u)(m)\right) + \left(m - \sum_{u \in E^0} \Phi(u)(m)\right),$$

and note that $\sum_{u \in E^0} \Phi(u)(m) \in \bigoplus_{u \in E^0} V_u$ and $\left(m - \sum_{u \in E^0} \Phi(u)(m)\right) \in \overline{V}$. So, it follows that

$$\left(\bigoplus_{u \in E^0} V_u\right) \oplus \overline{V} = V,$$

and the second part of the previous theorem applies to representations of any graph E , with E^0 finite (as long as (B2B) is satisfied).

Given a representation $\Phi : L_K(E) \rightarrow \text{Hom}_K(V)$, to see if this representation (or its restriction to W , Φ_1) is equivalent to a representation induced by an E -algebraic branching system, we must be able to, among other things, guarantee the existence of a basis of V satisfying the hypothesis of theorem 5.2. The existence of such a basis seems to be intrinsic to the representation Φ and to the module V , however, we prove in the next section that under a certain (sufficient but not necessary) condition over the graph E , it is always possible to choose such a basis of V .

6 A sufficient condition over E to guarantee equivalence of representations

Most of this section is inspired by corresponding results and ideas for graph C^* -algebras, as done in [5] and [6]. For the reader convenience, we adapt the necessary definitions and results below.

Definition 6.1. [6] Let $E = (E^0, E^1, r, s)$ be a graph. We say that:

1. A path without orientation between $u, v \in E^0$ is a pair of sequences $(u_0 u_1 \dots u_n; e_1 \dots e_n)$ of vertices u_i and edges e_j such that: $u = u_0$, $v = u_n$, $e_i \neq e_j$ for $i \neq j$, and for each i it holds that $s(e_i) = u_{i-1}$ and $r(e_i) = u_i$, or $r(e_i) = u_{i-1}$ and $s(e_i) = u_i$.
2. A graph E is P -simple if for each $u, v \in r(E^1) \cup s(E^1)$, with $u \neq v$, there exists at most one path without orientation between u and v , and moreover it does not exist $e \in E^1$ such that $r(e) = s(e)$.
3. Let $E = (E^0, E^1, r, s)$ be a graph. We say that a subset Z of E^0 is connected if, for each $u, v \in Z$, there exists a path without orientation between u and v .

For a given graph E , E^0 is obviously not necessarily connected, but it is always possible to write

$$E^0 = \left(\bigcup_{i \in \Delta} Z_i \right) \dot{\cup} R,$$

where each Z_i is connected and R is the set of isolated vertices.

Definition 6.2. [6] A vertex $v \in E^0$ is an extreme vertex of E if $\#\{r^{-1}(v) \cup s^{-1}(v)\} = 1$ and if there does not exist an edge $e \in E^1$ such that $r(e) = v = s(e)$. If v is an extreme vertex, then the unique edge adjacent to v is called an extreme edge.

We denote by X_1 the set of extreme vertices of E (the level 1 vertices) and by Y_1 the set of extreme edges of E (the level 1 edges). Notice that

$\mathbb{E}_1 = (E^1 \setminus Y_1, E^0 \setminus X_1, r, s)$ is a new graph (here r and s are the restriction maps $r, s : E^1 \setminus Y_1 \rightarrow E^0 \setminus X_1$). We denote by X_2 the set of extreme vertices of \mathbb{E}_1 (the level 2 vertices), and by Y_2 the extreme edges of \mathbb{E}_1 (the level 2 edges). Proceeding inductively we define the level n vertices set, X_n , and the level n edges set, Y_n , if such vertices and edges exist. For more details see [6].

Our aim in this section is to describe a sufficient condition, over the graph E , which guarantees that a representation of $L_K(E)$ is equivalent to a representation induced by an E -algebraic branching system. So, let us fix a representation $\Phi : L_K(E) \rightarrow \text{Hom}_K(V)$. Let $V_e := \Phi(e)\Phi(e^*)(V)$ and $V_u := \Phi(u)(V)$ for each $e \in E^1$ and $u \in E^0$. By theorem 5.2, all we need to do is verify the existence of basis of V_e and V_u satisfying the conditions of that theorem. Next we show the existence of such basis, under a certain condition over the graph E .

Theorem 6.3. *Let $E = (E^0, E^1, r, s)$ be a graph such that $r(E^1) \cup s(E^1)$ is connected and suppose*

$$r(E^1) \cup s(E^1) = \bigcup_{n=1}^m X_n \quad \text{or} \quad r(E^1) \cup s(E^1) = \bigcup_{n=1}^m X_n \cup \{\bar{v}\}.$$

Let $\Phi : L_K(E) \rightarrow \text{Hom}_K(V)$ be a representation. For each $e \in E^1$ and $v \in E^0$, consider the subspaces $V_e := \Phi(e)\Phi(e^)(V)$ and $V_u := \Phi(u)(V)$. Then, there exists basis B_e of V_e and B_u of V_u such that:*

- 1) *if $e \in s^{-1}(u)$, then $B_e \subseteq B_u$ and if $0 < |s^{-1}(u)| < \infty$, then $B_u = \bigcup_{e \in s^{-1}(u)} B_e$;*
- 2) *if $e \in r^{-1}(u)$, then $\Phi(e)(B_u) = B_e$. (and hence the basis satisfies hypothesis (B2B)).*

Proof. The proof is similar to the proof of [6:Theorem 4.1]. □

In the following proposition, we obtain a sufficient condition over the graph E to conclude that

$$r(E^1) \cup s(E^1) = \bigcup_{n=1}^m X_n \quad \text{or} \quad r(E^1) \cup s(E^1) = \bigcup_{n=1}^m X_n \cup \{\bar{v}\}.$$

Proposition 6.4. [6] *If $r(E^1) \cup s(E^1)$ is finite and connected and if E is P -simple then $r(E^1) \cup s(E^1) = \bigcup_{n=1}^m X_n$ or $r(E^1) \cup s(E^1) = \left(\bigcup_{n=1}^m X_n\right) \dot{\cup} \{\bar{v}\}$.*

We are now ready to prove the main result of this section, and finally present the reader with the condition over the graph E that guarantees that a representation of $L_K(E)$ is equivalent to a representation induced by an E -algebraic branching system.

Theorem 6.5. *Let $E = (E^0, E^1, r, s)$ be a graph. Write*

$$E^0 = \left(\bigcup_{i \in \Delta} Z_i\right) \dot{\cup} R,$$

where each Z_i is connected and R is the set of isolated vertices.

Suppose $Z_i = \bigcup_{n=1}^{m_i} X_n$ or $Z_i = \bigcup_{n=1}^{m_i} X_n \cup \{\bar{v}_i\}$ for each $i \in \Delta$ (for example, if each graph $E_i := (r^{-1}(Z_i) \cup s^{-1}(Z_i), Z_i, r, s)$ is P -simple and Z_i is finite, see proposition 6.4).

Let $\Phi : L_K(E) \rightarrow \text{Hom}_K(V)$ be a representation and suppose $\Phi(e^)(\bar{V}_e) = 0$ for each $e \in E^1$. Then:*

- 1. The representation Φ_1 (the restriction of Φ to W) is equivalent to a representation induced by an E -algebraic branching system.*
- 2. If $\Phi(u)(\bar{V}) = 0$, for each $u \in E^0$, then Φ is equivalent to a representation induced by an E -algebraic branching system.*

Proof. Let $\Phi : L_K(E) \rightarrow \text{Hom}_K(V)$ be a representation. Define $V_e = \pi(e)\pi(e^*)(V)$ and $V_u = \pi(u)(V)$. Note that $V = \bigoplus_{i \in \Delta} \left(\bigoplus_{u \in Z_i} V_u\right) \oplus \bar{V}$, where \bar{V} is some submodule of V . Applying theorems 6.3 and theorem 5.2 to each graph $E_i := (r^{-1}(Z_i) \cup s^{-1}(Z_i), Z_i, r, s)$, we obtain E -algebraic branching

systems:

$$(X_i, \{D_v\}_{v \in Z_i}, \{R_e\}_{e \in s^{-1}(Z_i) \cup r^{-1}(Z_i)}, \{f_e\}_{e \in s^{-1}(Z_i) \cup r^{-1}(Z_i)})$$

and K -module isomorphisms $U_i : \bigoplus_{u \in Z_i} V_u \rightarrow M_i$, for each $i \in \Delta$.

Now, the first part of the theorem follows if we consider the representation induced by the E -algebraic branching system

$$\left(\bigcup_{i \in \Delta} X_i, \{R_e\}_{e \in E^1}, \{D_u\}_{u \in E^0}, \{f_e\}_{e \in E^0} \right),$$

and the K -module isomorphism

$$U : \bigoplus_{i \in \Delta} \left(\bigoplus_{u \in Z_i} V_u \right) \rightarrow \bigoplus_{i \in \Delta} M_i \text{ given by } U := \bigoplus_{i \in \Delta} U_i.$$

The second statement of the theorem follows if we consider the E -algebraic branching system

$$\left(\left(\bigcup_{i \in \Delta} X_i \right) \cup I, \{R_e\}_{e \in E^1}, \{D_u\}_{u \in E^0}, \{f_e\}_{e \in E^0} \right),$$

where I is some index set (with $I \cap X_i = \emptyset$ for each $i \in \Delta$) of a (Hammel) basis $\{m_x : x \in I\}$ of \overline{V} , and the K -module isomorphism

$$Q : V = \bigoplus_{i \in \Delta} \left(\bigoplus_{u \in Z_i} V_u \right) \oplus \overline{V} \rightarrow N$$

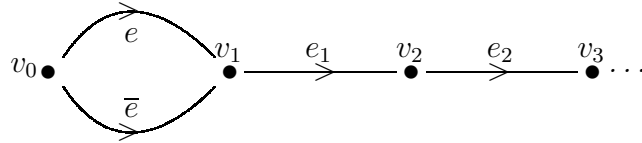
defined by $Q(m) = U(m)$, if $m \in \bigoplus_{i \in \Delta} \left(\bigoplus_{u \in Z_i} V_u \right)$, and $Q(m_x) = \delta_x$, if m_x is a element of the basis of \overline{V} . Notice that the K -module N is defined by

$$N = \left\{ g : \left(\bigcup_{i \in \Delta} X_i \right) \cup I \rightarrow K : g(x) \neq 0 \text{ only for finitely many } x \right\}.$$

□

The main idea of theorem 6.5 was to give a condition over the graph E that guarantees that the hypothesis of theorem 6.3 are satisfied. Below we give an example of a graph that does not satisfy the hypothesis of theorem 6.3, yet its conclusion (and hence the conclusion of theorem 6.5) is still valid.

Example 6.6. *Consider the graph*



Notice that this graph does not satisfy the hypothesis of theorem 6.3 or the hypothesis of proposition 6.4. However, given a representation $\Phi : L_K(E) \rightarrow \text{Hom}_K(V)$, it is possible to choose basis of $V_e := \Phi(e)\Phi(e^*)(V)$ and $V_u := \Phi(u)(V)$ satisfying $(B2B)$. Let us show how to choose such basis. First, fix a basis B_{v_1} of V_{v_1} . Recall that for each $e \in E^0$, $\Phi(e) : V_{r(e)} \rightarrow V_e$ and $\Phi(e^*) : V_e \rightarrow V_{r(e)}$ are K -module isomorphisms. So, $B_e := \Phi(e)(B_{v_1})$ is a basis of V_e and $B_{\bar{e}} := \Phi(\bar{e})(B_{v_1})$ is a basis of $V_{\bar{e}}$. Notice that $V_{e_i} = V_{v_i}$ for all $i \geq 1$. So, $B_{e_1} := B_{v_1}$ is a basis of V_{e_1} . Define $B_{v_2} := \Phi(e_1^*)(B_{e_1})$, which is a basis of V_{v_2} . Proceeding inductively, let $B_{e_i} := B_{v_i}$, which is a basis of V_{e_i} and define $B_{v_{i+1}} := \Phi(e_i^*)(B_{e_i})$. This way we obtain basis satisfying condition $(B2B)$. Following theorem 5.2, Φ is equivalent to a representation induced by an E -algebraic branching system.

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